Wave formation by time delays in randomly coupled oscillators

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We study the dynamics of randomly coupled oscillators when interactions between oscillators are time delayed due to the finite and constant speed of coupling signals. Numerical simulations show that the time delays, proportional to the Euclidean distances between interacting oscillators, can induce near regular waves in addition to near in-phase synchronous oscillations even though oscillators are randomly coupled. We discuss the stability conditions for the wave states and the in-phase synchronous states.

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Populations of coupled oscillators have been investigated as models of many physical, chemical, and biological systems [1–5]. In such models, time delays in interactions have been generally neglected, but recent studies show that time delays, comparable to the characteristic time scale of systems, can affect the dynamics significantly. Various effects such as multistability [6–10], amplitude death [11], clustering [12], and slow switching [13] have been identified through the studies of uniform time delays.

Recently, authors of Refs. [14,15] have reported that time delays can also induce traveling waves. In the framework of coupled oscillator systems without time delays, traveling waves are known as an emergent behavior of systems dominantly with short-range interactions [5], and not expected in systems with long-range interactions. Contrary to this, Refs. [14,15] have shown that time delay proportional to the Euclidean distances between interacting oscillators [16] can induce traveling waves in an array of coupled oscillators even with long-range interactions including all-to-all coupling.

These previous studies of time delays were carried out with regular topology of short-range [7,15] or long-range interactions [8–15]. In reality, however, such as in neuronal systems, coupling topologies are generally irregular [17]. Therefore, the studies need to be extended to the case of complex coupling topologies. In this direction, a study was reported recently on a common stability criterion for various coupling topologies when the time delays are uniform [18]. But for the case of distance-dependent time delays, it requires more investigations.

In this paper, as a starting point for studies of more complex coupling topology cases, we address the issue of the dynamic effect of distance-dependent time delays in a system of *randomly* coupled oscillators. One might expect that random coupling topology would yield random phase relationships between oscillators or possibly near in-phase synchronous oscillations but not support any regular structures. However, the results of our study shows, surprisingly, that distance-dependent time delays can induce near regular waves even though the oscillators are randomly coupled.

We consider the following system of coupled identical oscillators with time delays r_{ij}/v:

$$\dot{\theta}_i(t) = \omega_0 + \frac{K}{n_i}\sum_{j=1}^{N} A_{ij}\sin\left[\theta_j\left(t-\frac{r_{ij}}{v}\right)-\theta_i(t)\right],\ i=1,2,\ldots,N, \tag{1}$$

where $\theta_i(t)$ is the phase of ith oscillator at time t, ω_0 is the natural frequency of oscillators, and N is the total number of oscillators. The second term on the right side denotes the coupling between oscillator i and other oscillators. Oscillator i is coupled to n_i oscillators with coupling strength K according to a coupling topology described by an adjacency matrix A. Assuming bidirectional interaction between oscillators, we take the element of adjacency matrix $A_{ij}=A_{ji}=1$, if two oscillators i and j interact, and $A_{ij}=A_{ji}=0$ otherwise.

The coupling between oscillators i and j is assumed to be mediated by signal propagating the distance r_{ij} between the oscillators with constant speed v. The finite speed of signal causes the time delay $\tau_{ij}=r_{ij}/v$.

The oscillators are located on a ring with circumference L. The distance r_{ij} is unambiguously given by the shorter Euclidean distance between oscillators i and j *along* the ring [16]: $r_{ij}=\min\{|x_j-x_i|, L-|x_j-x_i|\}$, where x_i is the position of ith oscillator counterclockwise relative to a certain reference point on the ring. Note that in contrast to the cases of no time delay or uniform time delay, the positions of oscillators are important with this type of time delays. Here, we mainly consider the case of equal spacing between oscillators, in which oscillator i is located at $x_i=(L/N)i$. We will briefly discuss the case of random positioning of oscillators at the end of this paper.

The random coupling topologies discussed in this paper are constructed as follows. For each oscillator i, we choose randomly $\overline{n}/2$ oscillators which have no coupling with the oscillator i yet, and couple them bidirectionally to the oscillator i. As a result, $\overline{n}$ is just the average number of oscillators coupled to an oscillator; $\overline{n}=N^{-1}\Sigma_{i=1}^{N}n_i$. In this setting, the maximum distance between coupled oscillators is $L/2$ and thus the maximum time delay would be $L/2v$.

We set the natural frequency $\omega_0=\pi/10$ (the period $T=2\pi/\omega_0=20$), the number of oscillators $N=400$, and the circumference of the ring $L=400$. Due to the multistabilities usually observed in time-delayed systems [6–10,14,15], we need to consider various histories of the system for $t<0$ to

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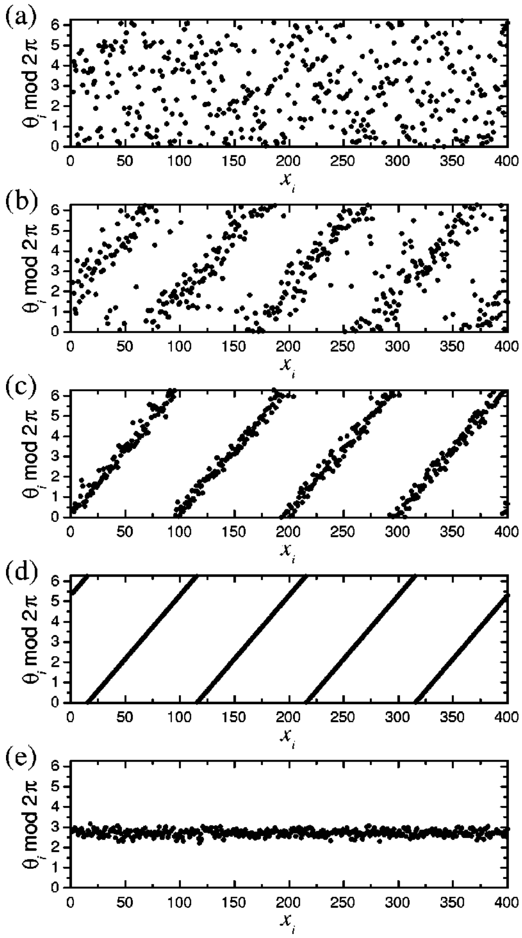

FIG. 1. Phase of the oscillators along the ring. $\omega_0 = \pi/10$, $N = L = 400$, $1/v = 0.22$, and $K = 0.4$. x_i is the position of the ith oscillator. (a) The average number of couplings per oscillator, $\overline{n} = 4$: no explicit structure, average frequency $\Omega \approx 0.31$, and frequency dispersion $\sigma \approx 0.05$. (b) $\overline{n} = 8$: a wave state with winding number $m = 4$, $\Omega \approx 0.295$, and $\sigma \approx 0.015$. (c) $\overline{n} = 20$: a wave state with $m = 4$, $\Omega = 0.2933$, and $\sigma = 4 \times 10^{-5}$. (d) All-to-all coupling case, $\overline{n} = N - 1 = 399$: a wave state with $m = 4$, $\Omega = 0.291\,15$, $\sigma < 10^{-5}$. (e) $\overline{n} = 20$: a near in-phase synchronous state, $\Omega = 0.0424$, and $\sigma < 10^{-5}$.

see the stable states. We take $\theta_i(t) = \Omega_{init} t + \phi_{i0}$, $i = 1, 2, \ldots, N$ for $t < 0$, where Ω_{init} is a common frequency and ϕ_{i0} is a constant phase for each i, and use various combinations of Ω_{init} and $\{\phi_{i0}\}$.

In the absence of time delays, for any positive values of the coupling strength K, this system typically exhibits in-phase synchronous oscillations with synchronization frequency equal to the natural frequency ω_0.

In contrast, in the presence of time delays, we observe that the system shows very different behaviors. Figure 1 shows the phase of the oscillators as a function of the position along the ring after almost stationary states are reached. The unit time delay $1/v = 0.22$ and the coupling strength $K = 0.4$ are used for all Figs. 1(a)–1(e). The differences in Figs. 1(a)–1(e) are due to the coupling topology and the history of oscillators. For random graphs with small average number of couplings per oscillator, $\overline{n} \leqslant 4$, the phases do not show any explicit structure [Fig. 1(a)]. On the other hand, for random graphs with more larger $\overline{n} \geqslant 8$, depending on the history of oscillators, the phase of oscillators can show almost linear change along the ring [Figs. 1(b) and 1(c)]. That is, the randomly coupled oscillators can exhibit almost regular traveling waves with a well-defined wavelength and an oscillating frequency. This regular wave formation in randomly coupled oscillators is an unexpected result showing the significant role of time delays, since it is generally accepted that randomly coupled elements cannot support any regular structures.

In addition to such states shown in Figs. 1(a)–1(c), for some range of parameters, randomly coupled oscillators with time delays can also exhibit near in-phase synchronous states, depending on the history of oscillators. For example, the wave state of Fig. 1(c) and the near in-phase state of Fig. 1(e) are multistable states of the same system with the same parameters. We also observe multistabilities of an in-phase state and wave states, and multistabilities of different wave states. These multistabilities are essentially the same with those observed in all-to-all coupled oscillators on one-dimensional space [14] and regularly coupled oscillators with finite interaction radius on two-dimensional space [15].

To quantify the oscillating behaviors of the system, we measure the average frequency Ω of oscillators defined by $\Omega \equiv \langle \dot{\theta} \rangle \equiv (1/N) \Sigma_i^N \dot{\theta}_i$ and the dispersion σ of frequency distribution defined by $\sigma \equiv \sqrt{\langle (\dot{\theta} - \Omega)^2 \rangle} = \sqrt{(1/N) \Sigma_i^N (\dot{\theta}_i - \Omega)^2}$. The smallness of the dispersion σ of a state represents that the state approaches to a frequency synchronized state where all the oscillators oscillate with the same frequency Ω. The states in Figs. 1(b)–1(e) have relatively small dispersion σ compared to the average frequency Ω, and thus we can say that the systems exhibit nearly frequency synchronized oscillations. This frequency synchronization was reported also in the cases of regularly coupled oscillators with distance-dependent time delays [14,15]. It is observed that as $\overline{n}$ of random graph increases [Fig. 1(a) $\rightarrow$ 1(b) $\rightarrow$ 1(c).], the state becomes more like that of all-to-all coupling case [Fig. 1(d)] [14]; the synchronization frequency Ω and the relative phases $\{\phi_i\}$ approach to those of all-to-all coupling case. Therefore, we can regard the random coupling case as an approximation of all-to-all coupling case.

In the remaining part of the paper, we focus on the wave forming systems. In numerical simulations, we take the case of random coupling topology with $N = 400$ and $\overline{n} = 20$, and the case of all-to-all coupling with $N = 400$.

To find out the synchronization frequency Ω and the relative phases $\{\phi_i\}$ characterizing the near frequency synchronized states of randomly coupled oscillators, we write the near frequency synchronized solution approximately as $\theta_i(t) = \Omega t + \phi_i$, where Ω and ϕ_i are constant. Substitution of this solution into Eq. (1) yields

$$\Omega \approx \omega_0 + \frac{K}{n_i} \sum_{j=1}^{N} A_{ij} \sin(-\Omega r_{ij}/v + \phi_j - \phi_i). \qquad (2)$$

This equation gives the possible synchronization frequency Ω for the given parameters and relative phases $\{\phi_i\}$.

For the state of randomly coupled oscillators represented by relative phases $\{\phi_i\}$ and a synchronization frequency Ω, the second term of Eq. (2) can be regarded as a randomly sampled value with sampling number n_i from the corresponding term of all-to-all coupled oscillators. This implies that, for the large enough n_is, the states of random coupled oscillators are approximately equal to those of all-to-all coupled oscillators. From this fact, we can say that since wave states are solutions of all-to-all coupled oscillators [14], randomly coupled oscillators can also have wave states. Note, however, that the stability of the states of all-to-all coupled oscillators does not guarantee the stability of corresponding states of randomly coupled oscillators.

For near in-phase synchronous oscillations or traveling waves, the relative phase ϕ_i can be written as $\phi_i \approx 2m\pi(x_i/L)+\phi_0$, where m is a winding number and ϕ_0 is a constant phase. Equation (2) becomes

$$\Omega \approx \omega_0 + \frac{K}{n_i}\sum_{j=1}^{N} A_{ij}\sin\left(-\Omega r_{ij}/v + 2m\pi\frac{x_j - x_i}{L}\right)$$
$$\approx \omega_0 + \frac{K}{N}\sum_{j=1}^{N} \sin\left(-\Omega r_{ij}/v + 2m\pi\frac{x_j - x_i}{L}\right). \qquad (3)$$

Figure 2(a) shows the synchronization frequency Ω as a function of unit time delay $1/v$ and winding number m for a fixed coupling strength $K=0.4$. The symbols denote the synchronization frequencies of stable states obtained from numerical simulations of Eq. (1). The curves for synchronization frequency Ω are obtained from Eq. (3), and fit well with the synchronization frequencies of randomly coupled oscillators (filled symbols) and all-to-all coupled oscillators (open symbols), respectively. As expected, the frequencies of random coupling case typically coincide with those of all-to-all coupling case for the corresponding stable states. But the stability regions for the random coupling case are smaller and within the stability regions of the all-to-all coupling case. The stability regions shrink more and more, as the average number $\overline{n}$ of couplings per oscillator decreases. These results confirm the previous observation that random coupling case approximates all-to-all coupling case more closely with larger $\overline{n}$.

Let us now discuss the stability of the frequency synchronized states. It is difficult to determine analytically the stability region for the states of the system with this type of time delays, because it requires studying infinitely many eigenvalues [14]. In the previous analytical studies of coupled oscillators with uniform time delay τ, the stability of frequency synchronized states—in these cases, in-phase synchronous states—is determined by the synchronization frequency Ω, time delay τ, and coupling strength K [6,9,18]. Especially, a quantity $\Omega\tau$ that can be interpreted as the virtual phase difference between each oscillator and the group of delayed oscillators affecting the oscillator has been recognized as an important factor in analyzing the stability of in-phase synchronous states [6,9,18]. Similarly, in the case of two-dimensional array of coupled oscillators with finite interaction radius r_0 and time delays r/v, in-phase synchronous states are shown to be stable only when a quantity $\Theta \equiv \Omega r_0/v$, which is related to the virtual phase difference, is less than a certain value Θ_c [15].

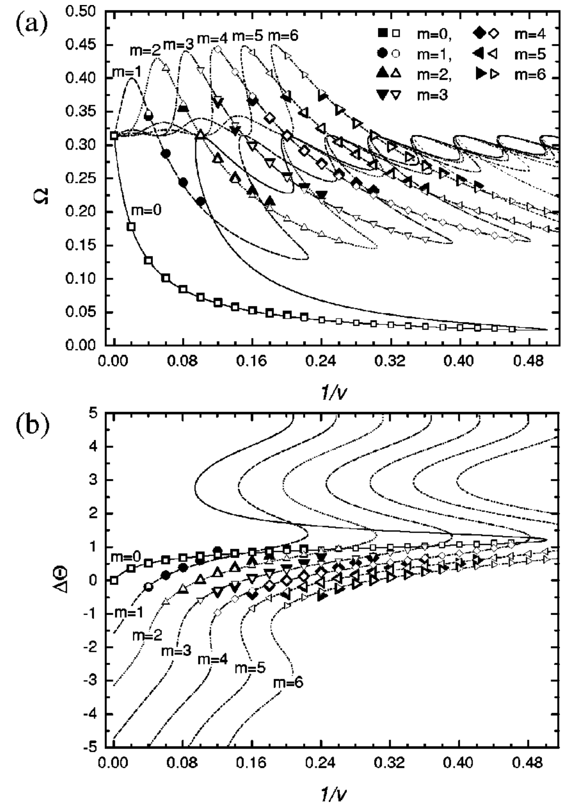

FIG. 2. (a) Synchronization frequency Ω as a function of a unit time delay $1/v$ and the winding number m of a state. $\omega_0=\pi/10$, $L=N=400$, and $K=0.4$. The filled symbols and the open symbols denote the stable states numerically obtained from Eq. (1) in random coupling cases with $\overline{n}=20$, and in all-to-all coupling case, respectively. Curves are obtained from Eq. (3). (b) Quantity $\Delta\Theta =(\Omega L/2v - m\pi)/2$ for the states of (a). Curves with $|\Delta\Theta|>5$ are not shown.

Likewise, we want to see the relevance of the virtual phase difference in determining stability of frequency synchronized states. In all-to-all coupled oscillators, the mean delayed oscillation of oscillators affecting oscillator i can be written as follows:

$$R(i,t,1/v)e^{\mathrm{i}\Phi(i,t,1/v)} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j(t-r_{ij}/v)}, \qquad (4)$$

where R and Φ are the amplitude and the phase of mean delayed oscillation, respectively. This mean delayed oscillation has the form of the delayed version of the complex order parameter of all-to-all coupled oscillators without time delays [1]. To get information of the virtual phase difference $[\theta_i(t)-\Phi(i,t,1/v)]$ in the frequency synchronized states, we compute $R\ \exp[\mathrm{i}(\theta_i-\Phi)]$ using integration for the solutions $\theta_i(t)=\Omega t+2m\pi(x_i/L)$. We get i-independent formula

$$R\ e^{\mathrm{i}(\theta_i-\Phi)} \approx \left[\frac{1}{2\Delta\Theta + 2m\pi} + \frac{1}{2\Delta\Theta}\right]\sin(\Delta\Theta)e^{\mathrm{i}\Delta\Theta}, \qquad (5)$$

where $\Delta\Theta=(\Omega L/2v - m\pi)/2$ and this $\Delta\Theta$ is just the virtual phase difference.

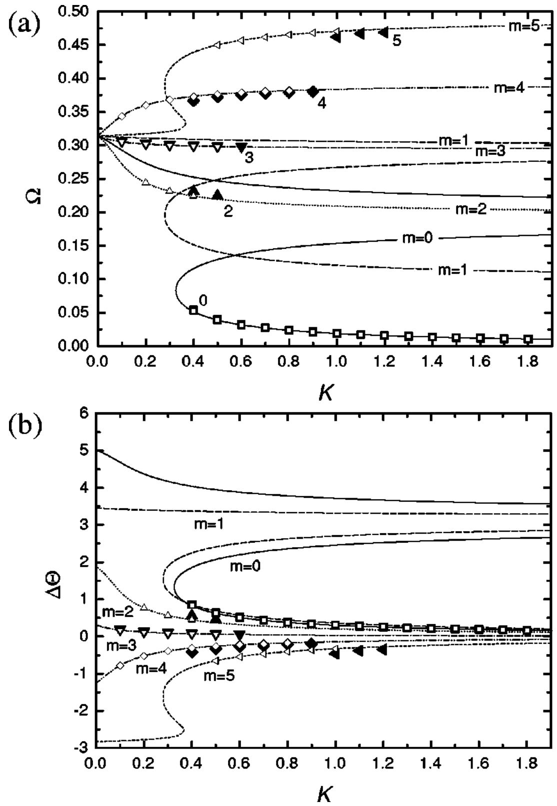

FIG. 3. (a) Synchronization frequency Ω as a function of coupling strength K and the winding number m of a state. Unit time delay $1/v=0.16$, and other parameters are the same as in Fig. 2. Symbols and letters are also used as in Fig. 2. Curves are obtained from Eq. (3). (b) Quantity $\Delta\Theta=(\Omega L/2v-m\pi)/2$ for the states of (a) as a function of coupling strength K.

Figure 2(b) shows the relevance of this virtual phase difference $\Delta\Theta$ in determining the stability of states. We compute the quantity $\Delta\Theta$ for each state depicted by Ω and m in Fig. 2(a). Note that the value increases monotonically along the curve for each m and the values for the stable states are confined into a region with $|\Delta\Theta|<\Theta_c\approx 1$. Frequency synchronized states with values outside of this region are observed to be unstable. Therefore, the virtual phase difference $\Delta\Theta$ can be a rough measure to determine the stability of the states. The condition $|\Delta\Theta|<\Theta_c$ summarizes the behaviors of the stable states of the system shown in Fig. 2(a). With a fixed m, Ω decreases as the unit time delay $1/v$ increases, and with fixed unit time delay $1/v$, Ω increases as m increases. For the same Ω, m increases as $1/v$ increases. This condition also implies that the velocity $v_p=\Omega L/2m\pi$ of the stable waves is approximately equal to the speed v of the coupling signal. As we can see in the following, this condition, however, is a sufficient one for the stability of in-phase states and a necessary one for the stability of wave states.

Now, we fix the unit time delay $1/v$ and see the effect of coupling strength K. Figure 3(a) shows the relationships between the synchronization frequency Ω, winding number m, and coupling strength K when the unit time delay is fixed to $1/v=0.16$. When K is smaller than a certain K_c, in this case, $K_c\sim 0.4$, only wave solutions are stable. For any coupling strength $K\geqslant K_c$, in-phase solutions ($m=0$) are stable. In contrast, wave states with winding number m are stable only for the coupling strength K between $K_m^{c_1}$ and $K_m^{c_2}$. Near the boundary of the ranges, wavelike states with large frequency dispersion σ comparable to the average frequency arise. As we increase the strength K, the wave solutions with higher winding number m and higher synchronization frequency become stable. The existence of a finite threshold coupling strength K_c in this system with time delays reflects the desynchronizing effect of time delays.

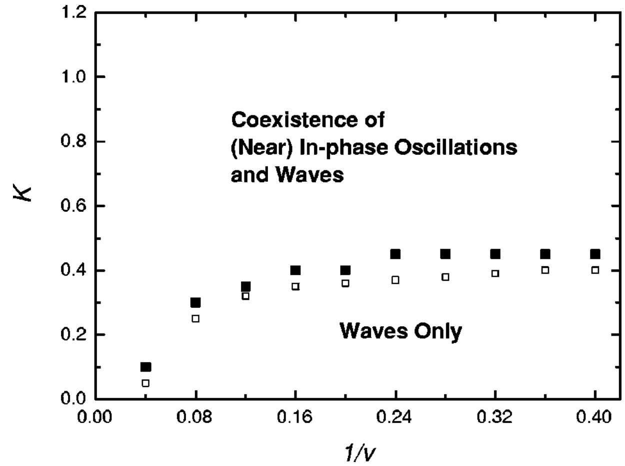

FIG. 4. Phase diagram of the model. $\omega_0=\pi/10$ and $L=N=400$. Below the symbols, (near) in-phase oscillations are not observed. The filled symbols denote the boundary for the random coupling cases with $\overline{n}=20$ and open symbols for all-to-all coupling case.

In Fig. 3(b), we plot the quantity $\Delta\Theta=(\Omega L/2v-m\pi)/2$ for the states of Fig. 3(a) as a function of coupling strength K. One of the branches of each solution approaches to zero as the coupling strength K increases. As mentioned before, the values for the stable states are confined into a region with $|\Delta\Theta|<\Theta_c\approx 1$. We observe that all the in-phase states satisfying the condition are stable. But, in the case of wave states, satisfying the condition does not guarantee the stability of the states. Therefore, we can conclude that the condition $|\Delta\Theta|<\Theta_c\approx 1$ is a sufficient condition for the stability of in-phase states and a necessary condition for the stability of wave states.

Figure 4 provides the phase diagram of the model as a function of unit time delay $1/v$ and coupling strength K. If the coupling strength K is below K_c which depends on the unit time delay $1/v$, the system has only wave states. With K above K_c, in-phase synchronous states in addition to the wave states are possible. The fact that K_c increases as $1/v$ increases shows the fact that more strong coupling is needed to overcome the larger desynchronizing effect of the time delays.

Finally, we check some details in this analysis. We simulate more sparsely coupled oscillators with $N=5000$ and $\overline{n}=20$ and obtain qualitatively the same results. Locating N oscillators on the sites randomly selected from $L/\Delta x(\geqslant N)$ sites, which are spaced $\Delta x=0.02$ apart, does not alter the results appreciably. We also observe the time-delay induced regular wave formation in the two-dimensional version of the model. These facts imply that the regular wave formation induced by time delay can occur in general settings.

In summary, we have investigated the dynamics of randomly coupled identical oscillators with time-delayed interactions mediated by signals of finite and constant speed. We have found that time delays proportional to the Euclidean distances between interacting oscillators can induce near regular waves in addition to near in-phase oscillations even in randomly coupled oscillators. Formation of such regular waves in randomly coupled oscillators is striking, since it has been generally accepted that randomly coupled elements cannot support any regular structures. We note that the wave formation by time delay does not come from any kind of mode instabilities [19]. Rather, it comes from the fact that the time delay is distance dependent and thus can carry geometrical information. This makes regular wave formation possible even when the coupling topology can otherwise prevent the formation of any regular structures. We have also found that the virtual phase difference between each oscillator and the group of delayed oscillators affecting the oscillator is crucial for the stability of in-phase states and wave states. Our results may be helpful to understand wave formation in neuronal systems with very complex coupling topologies and nonlocal connections [5].

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